§11.7. Strategy for Testing Series

1. Three basic types of series and one rule.

① p-Series. $\sum \frac{1}{n^p}$ $\begin{cases} \text{ConV} & \text{if } p>1 \\ \text{DIV} & \text{if } p\le 1 \end{cases}$

② Geometric Series: $\sum a\cdot r^{n-1}$ (or $\sum a\cdot r^n$, $\sum a\cdot r^{n+1}$, etc) $\begin{cases} \text{ConV} & \text{if } |r|<1 \quad \left(=\frac{a}{1-r}\right) \\ \text{DIV} & \text{if } |r|\ge 1 \end{cases}$

③ Alternating Series: $\sum (-1)^n \cdot b_n$ (or $\sum(-1)^{n-1} b_n$, $\sum (-1)^{n+1} b_n$, etc) $\begin{cases} \text{ConV} & \text{if } \lim_n b_n = 0 \text{ and } b_n \text{ decreases.} \\ \text{DIV} & \text{if } \lim b_n \ne 0 \end{cases}$

One rule: n^{th} term test for divergence.

④ If $\lim a_n \ne 0$, then $\sum a_n$ is divergent.

2. Check whether ①-④ can be applied directly or can be applied through some simple simplification (of a_n).

e.g. Test for ConV/DIV.

- $\sum 3\cdot n^{-\frac{3}{2}}$ p-Series $p=\frac{3}{2}$, ConV
- $\sum \frac{3^n}{2^{2n}}$ G.S. $r=\frac{3}{4}$, ConV.
- $\sum (-1)^{n+1} \cdot \frac{1}{\sqrt{n}}$ A.S. $b_n = \frac{1}{\sqrt{n}}$. conV
- $\sum (-1)^n \cdot e^{\frac{1}{n}}$ DIV Test, $\lim_n e^{\frac{1}{n}} = e^0 = 1 \ne 0$.

- $\sum \left(5^n + \frac{1}{n^2}\right) = \sum 5^n + \sum \frac{1}{n^2}$
  DIV + ConV $\Rightarrow$ DIV
- $\sum \frac{n+1}{n^3} = \sum \frac{1}{n^2} + \sum \frac{1}{n^3}$
  conV + conV $\Rightarrow$ conV.
- $\sum \frac{3^n + 2^n}{4^n} = \sum \left(\frac{3}{4}\right)^n + \sum \left(\frac{2}{4}\right)^n$
  conV + conV $\Rightarrow$ conV.

3. If none of ①-④ works, then consider whether a_n can be compared with p-Series or G.S. And try to apply (limit) Comparison Test.

- $\sum \frac{1-6n+3n^2}{n^4+1}$ compare with $\sum \frac{3n^2}{n^4}$. ConV.
- $\sum \frac{2^n+1}{3^n-1}$ compare with $\sum \frac{2^n}{3^n}$ conV. via Limit Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

4. If a_n is not so clear to be compared with some known series, then try Ratio Test.

In particularly, if a_n contains factorial $n!$ or the problem asks to test ABS convergence, then consider Ratio Test first.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} L<1, & \sum a_n \text{ is ABS convergent and also convergent} \\ L>1, & \sum a_n \text{ is divergent} \\ L=1. & \text{R. Test inconclusive. (Then try Alternating S. Test or Definition of ABS conv)} \end{cases}$$

e.g. $a_n = \frac{3^n \cdot n^2}{n!}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}\cdot(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n\cdot n^2} = \frac{3^{n+1}}{3^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n!}{(n+1)!} = 3\cdot\frac{(n+1)^2}{n^2}\cdot\frac{1}{n+1} = \frac{3(n+1)}{n^2}$

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3(n+1)}{n^2} = 0 \ (<1)$. $\Rightarrow \sum\frac{3^n n^2}{n!}$ conv.

5. If 2-4 do not work, then try Integral Test. In particularly, for series $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^p\cdot n}$, use Integral Test with $\int_2^{\infty}\frac{1}{(\ln x)^p\cdot x}dx$ (p can be positive or negative)

e.g. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{\ln n}\cdot n} \overset{\text{I. Test}}{\sim} \int_2^{\infty}\frac{1}{\sqrt{\ln x}\cdot x}dx = \lim_{t\to\infty}\int_2^t\frac{1}{\sqrt{\ln x}\cdot x}dx$, $u=\ln x$, $du=\frac{1}{x}dx$.

$= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{\sqrt{u}}du = \lim_{t\to\infty}2\sqrt{u}\Big| = \lim_{t\to\infty}2\sqrt{\ln x}\Big|_2^t$

$= \lim_{t\to\infty}2\sqrt{\ln t}-2\sqrt{\ln 2} = \infty$

DIV since $\int_2^{\infty}\frac{1}{\sqrt{\ln x}\cdot x}dx = \infty$.

6. Conclusion:

Order to chose tests: $\left.\begin{array}{l}\text{DIV Test}\\ \text{p-Series}\\ \text{Geo. Series}\\ \text{Alternating}\end{array}\right\} \longrightarrow$ (limit) Comparison $\to$ Ratio Test $\to$ Integral Test.

Remark 1: Above tests are used to determine whether $\sum a_n$ is convergent or DIV. They can not be used to compute the exact value of $\sum a_n$ EXCEPT Geometric Series (Test). In other words, once you see the key words "Find the sum" "Compute the series" etc, try to use the formula $\sum_{n=0}^{\infty}a\cdot r^n = \frac{a}{1-r}$, $|r|<1$

Remark 2: If a_n are all positive (or all negative), ABS conv and conv are the same. They are different only for alternating series $\sum(-1)^n\cdot b_n$ and ABS conv $\Rightarrow$ conv. (Not vice verse)